NOTE ON THE STABILITY OF MEMBRANE SOLUTIONS IN THE NONLINEAR THEORY OF PLATES AND SHELLS

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Membrane solutions in the nonlinear theory of the equilibrium equations for shells and plates i.e. solutions corresponding to the state of str ess under the action of tensile forces, only, were considered in references [1 and 2]. The uniqueness of the solutions was shown, a series of existence theorems were given, and in addition the asymptotic solution in the case of small stiffness was studied in [1 and 2].

The present note considers a plate of arbitrary shape under normal load and (nonlinear) forces on the contour, and a shell clamped along the edge and under the action of external forces, including those radially symmetric.

It is shown that every membrane solution of these problems results in the potential energy minimum (second variation of the energy is positive) and is therefore stable. In particular, it is established that every solution of a corresponding nonstationary problem, which, in the initial moment in an energy norm in H_Ω (see equation (3.2) below) resembles the membrane solution, does so for all values of $t > 0$; i.e. the membrane solution is stable in the Liapunov sense in the energy norm. It is also shown that alternation in the sign of the second variation of energy results in the instability. A series of examples is given.

1. Let us consider a system of nonlinear von Kàrmàn equations for flexible plates [3]

$$\Delta^2 F + w_{xx}w_{yy} - w_{xy}^2 = 0 \qquad (1.1)$$

$$\varepsilon^2\Delta^2 w - w_{xx}F_{yy} - w_{yy}F_{xx} + 2w_{xy}F_{xy} - q = 0 \qquad (1.2)$$

with the edge conditions

$$w|_\Gamma = 0, \qquad w_n|_\Gamma = 0 \qquad (1.3)$$

$$F_{\tau\tau}|_\Gamma = T(A) \geqslant 0, \qquad F_{n\tau}|_\Gamma = S(A) \qquad (A \in \Gamma) \tag{1.4}$$

Equations (1.1) to (1.4) are written in dimensionless form. At the same time

$$F = \frac{F_1}{Ea^2}, \quad w = \frac{w_1}{a}, \quad q = \frac{q_1 a}{Eh}, \quad \varepsilon^2 = \frac{h^2}{12(1-\mu^2)a^2}$$
$$x = \frac{x_1}{a}, \quad y = \frac{y_1}{a}, \quad n = \frac{n_1}{a}, \quad \tau = \frac{\tau_1}{a} \qquad \left(0 < \mu < \frac{1}{2}\right)$$

Here F_1 is a stress function, w_1 is the deflection of points on the median surface, q_1 is the transverse load intensity, h is the plate thickness, E is Young's Modulus, μ is the Poisson's ratio, (x_1, y_1) denote rectangular coordinates, Γ is the boundary of a simply connected region Ω, a is the diameter of this region, n_1 and τ_1, normal and tangential boundary stresses respectively, and $F_{\tau\tau}(A)$ and $F_{n\tau}(A)$, normal and tangential components of the external forces applied at the plate contour [3].

The functional of the potential energy of the plate corresponding to the problem (1.1) to (1.4) may be written in the form

$$\begin{aligned} J(w) = \frac{\varepsilon^2}{2}\int_\Omega [(\Delta w)^2 - 2(1-\mu)(w_{xx}w_{yy} - w_{xy}^2)]\,dx\,dy - \\ - \frac{1}{2}\int_\Omega [(\Delta F)^2 - 2(1+\mu)(F_{xx}F_{yy} - F_{xy}^2)]\,dx\,dy + \\ + \frac{1}{2}\int_\Omega [F_{xx}w_y^2 + F_{yy}w_x^2 - 2F_{xy}w_xw_y]\,dx\,dy - \int_\Omega qw\,dx\,dy \end{aligned} \tag{1.5}$$

Then every solution of the problem (1.1) to to (1.4) results in the minimum of the functional J with respect to the function w satisfying the boundary conditions (1.3). Here, the function F is considered as a solution of (1.1) with the boundary conditions (1.4). Let (Φ, W) be the membrane solution of the problem (1.1) to (1.4), i.e. let the conditions

$$\Phi_{xx} > 0, \qquad \Phi_{yy} > 0, \qquad \Phi_{xx}\Phi_{yy} - \Phi_{xy}^2 > 0 \tag{1.6}$$

be satisfied at every point of the region Ω.

We shall show that the second variation of the functional J is positive for the membrane solution. Let us denote by η the allowed variation in w and consider the functional J on the set of functions $W + \alpha\eta$. Then F should be a solution of the boundary problem (1.1) and (1.4). It is easy to see that F may be represented in the form

$$F = \Phi + \alpha\varphi + \alpha^2\psi \tag{1.7}$$

where Φ satisfies an equation of the type (1.1) but with w substituted for W and with the boundary conditions (1.4); the function φ satisfies the equation

$$\Delta^2\varphi = 2W_{xy}\eta_{xy} - W_{xx}\eta_{yy} - W_{yy}\eta_{xx}, \quad \varphi|_\Gamma = 0, \quad \varphi_n|_\Gamma = 0 \tag{1.8}$$

while the function ψ satisfies the equation

$$\Delta^2\psi = \eta_{xy}^2 - \eta_{xx}\eta_{yy}, \qquad \psi|_\Gamma = 0, \qquad \psi_n|_\Gamma = 0 \tag{1.9}$$

Equations (1.7) to (1.9) are obtained from (1.1) to (1.4) by substituting $W + \alpha\eta$ for w. Calculating $\delta^2 J$, we obtain

$$\delta^2 J = \frac{1}{2}\frac{d^2 J}{d\alpha^2}\bigg|_{\alpha=0} = \varepsilon^2 I_1(\eta) - I_2(\varphi) + I_3(\Phi, \eta) + I_3(\psi, W) - I_4 + I_5 \tag{1.10}$$

$$I_1(\eta) = \frac{1}{2}\int_\Omega \{(\Delta\eta)^2 - 2(1-\mu)(\eta_{xx}\eta_{yy} - \eta_{xy}^2)\}\, dx\, dy$$

$$I_2(\varphi) = \frac{1}{2}\int_\Omega \{(\Delta\varphi)^2 - 2(1+\mu)(\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2)\}\, dx\, dy$$

$$I_3(a, b) = \frac{1}{2}\int_\Omega (a_{xx}b_y^2 + a_{yy}b_x^2 - 2a_{xy}b_x b_y)\, dx\, dy \tag{1.11}$$

$$I_4 = \int_\Omega \{\psi_{yy}(\Phi_{yy} - \mu\Phi_{xx}) + \psi_{xx}(\Phi_{xx} - \mu\Phi_{yy}) + 2(1+\mu)\psi_{xy}\Phi_{xy}\}\, dx\, dy$$

$$I_5 = \int_\Omega \{\varphi_{xx}W_y\eta_y + \varphi_{yy}W_x\eta_x - \varphi_{xy}(W_y\eta_x + W_x\eta_y)\}\, dx\, dy$$

Integrating by parts and taking the boundary conditions, (1.8) and (1.9) into account, we have

$$I_3(\psi, W) = \int_\Omega \psi\Delta^2\Phi\, dx\, dy = I_4$$

$$I_5 = \int_\Omega (\Delta\varphi)^2\, dx\, dy, \qquad \int_\Omega (\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2)\, dx\, dy = 0 \tag{1.12}$$

Assuming (1.5), we obtain from (1.10)

$$\delta^2 J = \varepsilon^2 I_1(\eta) + I_2(\varphi) + I_3(\Phi, \eta) \tag{1.13}$$

It should be noted that $I_3(\Phi, \eta) \geqslant 0$ by virtue of (1.6), and consequently

$$\delta^2 J > 0 \quad (\eta \not\equiv 0) \tag{1.14}$$

Hence the membrane solution results in the minimum of J.

2. At this stage, we shall consider the equations for a shallow shell occupying in its plane a region Ω with a boundary Γ [4]

$$\Delta u + \frac{1+\mu}{1-\mu}\theta_x = -\frac{2}{1-\mu}[(k_1 w)_x + w_x w_{xx} + \mu(k_2 w)_x + \mu w_y w_{xy}] - \\ - w_{xy}w_y - w_x w_{yy} = f_1 \qquad (\theta = u_x + v_y) \tag{2.1}$$

$$\Delta v + \frac{1+\mu}{1-\mu}\theta_y = -\frac{2}{1-\mu}[(k_2 w)_y + w_y w_{yy} + \mu(k_1 w)_y + \mu w_x w_{xy}] - \\ - w_{xy}w_x - w_y w_{xx} = f_2 \tag{2.2}$$

$$D\Delta^2 w = Z + F_{yy}(w_{xx} - k_1) + F_{xx}(w_{yy} - k_2) - 2F_{xy}w_{xy} \tag{2.3}$$

$$\begin{gathered} F_{yy} = \frac{Eh}{1-\mu^2}(\varepsilon_1 + \mu\varepsilon_2), \quad F_{xx} = \frac{Eh}{1-\mu^2}(\varepsilon_2 + \mu\varepsilon_1), \quad F_{xy} = -\frac{Eh}{2(1+\mu)}\varepsilon_{12} \\ \varepsilon_1 = u_x + k_1 w + {}^1/_2 w_x^2, \quad \varepsilon_2 = v_y + k_2 w + {}^1/_2 w_y^2, \quad \varepsilon_{12} = u_y + v_x + w_x w_y \end{gathered} \tag{2.4}$$

The shell is clamped along the edge and the boundary conditions are

$$w|_\Gamma = 0, \quad w_n|_\Gamma = 0, \qquad u|_\Gamma = 0, \quad v|_\Gamma = 0 \tag{2.5}$$

Every solution of (2.1) to (2.5) results in the minimum of the functional

$$\begin{gathered} J(w) = \frac{D}{2}\int_\Omega \{(\Delta w)^2 - 2(1-\mu)(w_{xx}w_{yy} - w_{xy}^2)\}\,dx\,dy + \\ + \frac{1}{2Eh}\int_\Omega \{F_{xx}^2 + F_{yy}^2 - 2\mu F_{xx}F_{yy} + 2(1+\mu)F_{xy}^2\}\,dx\,dy + \int_\Omega Zw\,dx\,dy \end{gathered} \tag{2.6}$$

with respect to the function w satisfying conditions (2.5). The functions F_{xx}, F_{yy}, and F_{xy} are expressed by formulas (2.4) with u and v considered as solutions of (2.1) and (2.2).

As in section 1, let (Φ, W) be the membrane solution of (2.1) to (2.5). Let us compute the second variation. To do this, we shall consider the functional J or set of functions $W + \alpha\eta$. Then F may be represented in the form (1.7). At the same time, Φ satisfies equations of the type (2.4), (2.1), (2.2) and (2.5), but with Φ substituted for F and w for W. The function φ satisfies the relations

$$\begin{aligned} \varphi_{yy} &= \frac{dF_{yy}}{d\alpha}\Big|_{\alpha=0} = \frac{Eh}{1-\mu^2}[u_{1x} + k_1\eta + W_x\eta_x + \mu(v_{1y} + k_2\eta + W_y\eta_y)] \\ \varphi_{xx} &= \frac{dF_{xx}}{d\alpha}\Big|_{\alpha=0} = \frac{Eh}{1-\mu_1}[v_{1y} + k_2\eta + W_y\eta_y + \mu(u_{1x} + k_1\eta + W_x\eta_x)] \\ \varphi_{xy} &= \frac{dF_{xy}}{d\alpha}\Big|_{\alpha=0} = -\frac{Eh}{2(1+\mu)}(u_{1y} + v_{1x} + W_x\eta_y + W_y\eta_x) \end{aligned} \tag{2.7}$$

Here the functions u_1 and v_1 are determined from the equations

$$\begin{gathered} \Delta u_1 + \frac{1+\mu}{1-\mu}\theta_{1x} = f_{11}, \qquad f_{11} = \frac{d}{d\alpha}f_1(W+\alpha\eta)|_{\alpha=0}, \qquad u_1|_\Gamma = 0 \\ \Delta v_1 + \frac{1+\mu}{1-\mu}\theta_{1y} = f_{21}, \qquad f_{21} = \frac{d}{d\alpha}f_2(W+\alpha\eta)\Big|_{\alpha=0}, \; v_1|_\Gamma = 0, \; \theta_1 = u_{1x} + v_{1y} \end{gathered} \tag{2.8}$$

Function ψ satisfies the relations

$$\begin{aligned} \psi_{xx} &= \frac{1}{2}\frac{d^2F_{xx}}{d\alpha^2}\Big|_{\alpha=0} = \frac{Eh}{1-\mu^2}\left(v_{2y} + \frac{1}{2}\eta_y^2 + \mu u_{2x} + \frac{\mu}{2}\eta_x^2\right) \\ \psi_{yy} &= \frac{1}{2}\frac{d^2F_{xy}}{d\alpha^2}\Big|_{\alpha=0} = \frac{Eh^1}{1-\mu^2}\left(u_{2x} + \frac{1}{2}\eta_x^2 + \mu v_{2y} + \frac{\mu}{2}\eta_y^2\right) \\ \psi_{xy} &= -\frac{1}{2}\frac{d^2F_{xy}}{d\alpha^2}\Big|_{\alpha=0} = -\frac{Eh}{1+\mu}(u_{2y} + v_{2x} + \eta_x\eta_y) \end{aligned} \tag{2.9}$$

Here the functions u_2 and v_2 are determined from the equations

$$\Delta u_2 + \frac{1+\mu}{1-\mu}\theta_{2x} = f_{12}, \quad \Delta v_2 + \frac{1+\mu}{1-\mu}\theta_{2y} = f_{22}, \quad f_{12} = \frac{1}{2}\frac{d^2}{d\alpha^2} f_1(W+\alpha\eta)\Big|_{\alpha=0}$$
$$f_{22} = \frac{1}{2}\frac{d^2}{d\alpha^2} f_2(W+\alpha\eta)\Big|_{\alpha=0}, \quad u_2|_\Gamma = 0, \quad v_2|_\Gamma = 0, \quad \theta_2 = u_{2x} + v_{2y} \tag{2.10}$$

Formulas (2.7) to (2.10) are obtained from (2.1) to (2.5) by substituting $W + \alpha\eta$ for w. Now, from (2.6), according to (1.10) and applying (1.7), we obtain

$$\delta^2 J = DI_1 + \frac{1}{Eh} I_2 + \frac{1}{Eh} I_4 \tag{2.11}$$

Using (2.9) and integrating by parts while taking account of boundary conditions (2.10) we obtain

$$I_4 = I_3(\Phi, \eta) \tag{2.12}$$

Therefore

$$\delta^2 J = DI_1(\eta) + \frac{1}{Eh} I_2(\varphi) + I_3(\Phi, \eta) \tag{2.13}$$

From this we obtain

$$\delta^2 J > 0 \qquad (\eta \not\equiv 0) \tag{2.14}$$

for the membrane solution.

Finally, we consider the case of a symmetrically loaded shell of revolution, clamped along the edge. We shall transform the coordinates in (2.1) to (2.5) into polar (r, φ) and introduce the dimensionless quantities

$$\rho = \frac{r}{a}, \quad u_0 = w_\rho, \quad v_0 = \frac{F_\rho}{ahE}, \quad \varphi_0(\rho) = \frac{a}{hE}\int_0^\rho q(t)\, t dt \tag{2.15}$$

For a symmetrical solution we obtain the equations [5]

$$Av_0 - \frac{u_0^2}{2} + \theta(\rho) u_0 = 0, \qquad A(\quad) \equiv -\rho \frac{d}{d\rho}\frac{h^2}{\rho}\frac{d}{d\rho}\rho(\quad) \tag{2.16}$$

$$\varepsilon^2 A u_0 + u_0 v_0 - \theta(\rho) v_0 + \varphi_0(\rho) = 0, \qquad \varepsilon^2 = \frac{h}{12(1-\mu^2)a^2} \tag{2.17}$$

$$u_0|_{\rho=1} = 0, \quad \left[\frac{dv_0}{d\rho} - \frac{\mu}{\rho} v_0\right]_{\rho=1} = 0, \quad \frac{u_0}{\rho}\Big|_{\rho=0} < \infty, \quad \frac{v_0}{\rho}\Big|_{\rho=0} < \infty \tag{2.18}$$

From (2.6) we find, by applying (2.15), the corresponding functional:

$$J(u_0) = \frac{\varepsilon^2}{2}\int_0^1 \left(\rho u_{0\rho}^2 + \frac{u_0^2}{\rho}\right) d\rho + \frac{1}{2}\int_0^1 \left(\rho v_{0\rho}^2 + \frac{v_0^2}{\rho}\right) d\rho - \frac{\mu}{2} v_0^2(1) + \int_0^1 \varphi_0(\rho) u_0\, d\rho \tag{2.19}$$

with respect to a function u_0 satisfying the boundary conditions (2.18). The function v_0 is considered to be a solution of equations (2.16) and (2.18).

Let us find the second variation of $J(u_0)$ for the membrane solution v_*, u_* (i.e. $v_* \geqslant 0$, see [1]). We consider the functional J on the set of functions $u_0 = u_* + \alpha\eta$.

Then analogously to section 1, v_0 may be represented in the form

$$v_0 = v_* + \alpha\varphi + \alpha^2\psi \tag{2.20}$$

where φ satisfies the equation

$$A\varphi - u_*\eta + \theta\eta = 0, \quad \left[\frac{d\varphi}{d\rho} - \frac{\mu}{\rho}\varphi\right]_{\rho=1} = 0, \quad \frac{\varphi}{\rho}\bigg|_{\rho=0} < \infty \tag{2.21}$$

and where ψ is the solution of the problem

$$A\psi - \frac{1}{2}\eta^2 = 0, \quad \left[\frac{d\psi}{d\rho} - \frac{\mu}{\rho}\psi\right]_{\rho=1} = 0, \quad \frac{\psi}{\rho}\bigg|_{\rho=0} < \infty \tag{2.22}$$

By (1.12), and using (2.21) and (2.22), we have

$$\delta^2 J = \frac{\varepsilon^2}{2}\int_0^1\left(\rho\eta_\rho^2 + \frac{\eta^2}{\rho}\right)d\rho + \frac{1}{2}\int_0^1\left(\rho\varphi_\rho^2 + \frac{\varphi^2}{\rho}\right)d\rho - \frac{\mu}{2}\varphi^2(1) + \frac{1}{2}\int_0^1 v_*\eta^2\,d\rho \tag{2.23}$$

Since $v_* \geqslant 0$, the last integral in (2.23) is positive and consequently $\delta^2 J > 0$.

3. To give more precise information on what should be understood by the word stability, we consider the nonstationary equations of the theory of shallow shells (formulas 2.5 to 2.8 and notation of [6], are used).

Let the second variation $\delta^2 J$ calculated for the stationary solution W (see 1.10) satisfy the condition

$$\delta^2 J \geqslant m\|\eta\|^2_{H_{2\Omega}} \qquad (m > 0) \tag{3.1}$$

Here η is an allowed variation of the displacement w. We note that in the above problem this estimate easily follows from the membrane solution of (1.10) and (2.11). Further, let

$$\|\eta_t\|^2_{H_{1\Omega}} + \|\eta\|^2_{H_{2\Omega}} = \|\eta\|^2_{H_\Omega} \tag{3.2}$$

and $w(x, y, t)$ be an arbitrary solution (though generalized) of the known nonstationary system, continuous in t for $t \geqslant 0$ as a function of t in H_Ω. Let $\varepsilon > 0$ be any number. Then such $\delta > 0$ can be found, that from the inequality

$$\|w(x, y, 0) - W(x, y)\|_{H_\Omega} < \delta \tag{3.3}$$

it follows that

$$\| w(x, y, t) - W(x, y) \|_{H_\Omega} < \varepsilon \qquad \text{for} \quad t > 0 \tag{3.4}$$

In other words, the stationary solution W is stable in the Liapunov sense in H_Ω.

To prove this we shall note that the potential energy $J[w(t)]$ may be presented in the form

$$J(W + \eta) = J(W) + \delta^2 J + R(\eta) \tag{3.5}$$

Here the functional $R(\eta)$ has the form

$$R(\eta) = \frac{1}{2} \int_\Omega (\varphi_{xx}\eta_y^2 + \varphi_{yy}\eta_x^2 - 2\varphi_{xy}\eta_x\eta_y)\, dx\, dy \tag{3.6}$$

where φ_{xx}, φ_{yy}, and φ_{xy} are defined in (2.7). We shall show that the estimate

$$|R(\eta)| \leqslant m_1 \|\eta\|^3_{H_{2\Omega}} \tag{3.7}$$

holds.

Let us consider the typical terms in the expression (3.5)

$$R_1(\eta) = \int_\Omega u_{1x}\eta_y^2\, dx\, dy, \qquad R_2(\eta) = \int_\Omega v_{1y}\eta_x^2\, dx\, dy$$

$$R_3(\eta) = \int_\Omega k_1\eta\eta_x^2\, dx\, dy, \qquad R_4(\eta) = \int_\Omega W_y\eta_y\eta_x^2\, dx\, dy \tag{3.8}$$

We have, by virtue of the insertion theorem [7]

$$\| W_y \|_{L_{p\Omega}} \leqslant m_2 \| W \|_{H_{2\Omega}}, \quad \| \eta_x \|_{L_{p\Omega}} \leqslant m_2 \| \eta \|_{H_{2\Omega}} \quad (p > 1) \tag{3.9}$$

Further, from (2.8) one may obtain

$$\| u_{1x} \|_{L_{2\Omega}} \leqslant m_3 \| \eta \|_{H_{2\Omega}}, \qquad \| v_{1y} \|_{L_{2\Omega}} \leqslant m_3 \| \eta \|_{H_{2\Omega}} \tag{3.10}$$

These estimates were obtained in [6] in formulas (2.41) to (2.45). From (3.8), applying the Buniakovski's inequality and utilising (3.9) and (3.10), we obtain

$$| R_i(\eta) | \leqslant m_4 \| \eta \|^3_{H_{2\Omega}} \qquad (i = 1, 2, 3, 4) \tag{3.11}$$

Now (3.7) follows from (3.11), and from similar estimates for the other terms in (3.6).

The considerations which follow, are analogous to the proof of the Liapunov stability theorem [8] *. The functional V determined by formula (3.16) plays the role of a Liapunov function.

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* Note that in Movchan's work [15 and 16] the theorems of Liapunov and Chetaev are applied to certain classès of infinite conservative systems.

We shall write the given nonstationary system in the form

$$w_{tt} = -\operatorname{grad} J(w) \tag{3.12}$$

Solutions of (3.12) satisfy the energy integral

$$\|w_t\|^2_{H_{1\Omega}} + 2J(w) = \text{const} \tag{3.13}$$

Let us put $\eta(x, y, t) = w(x, y, t) - W(x, y)$. From (3.5) and (3.13) it follows that

$$\|\eta_t\|^2_{H_{1\Omega}} + 2J(W) + 2\delta^2 J + 2R(\eta) = \text{const} \tag{3.14}$$

Hence

$$V_1(t) \equiv \|\eta\|^2_{H_{1\Omega}} + 2\delta^2 J(\eta) + 2R(\eta) = \tag{3.15}$$
$$= \|\eta_{0t}\|^2_{H_{1\Omega}} + 2\delta^2 J(\eta_0) + 2R(\eta_0) = V_1(0), \quad \eta_0 = \eta|_{t=0}$$

From (3.15) and (3.1) we obtain

$$V(t) \equiv c\|\eta\|^2_{H_\Omega} + R(\eta) < V_1(t) = V_1(0) \tag{3.16}$$

Here $c = \min(1, m)$, and $R(\eta)$ satisfies the inequality (3.7).

Let us consider a sphere S_ε in the space H_Ω

$$\|\eta\|^2_{H_\Omega} = \|\eta_t\|^2_{H_{1\Omega}} + \|\eta\|^2_{H_{2\Omega}} = \varepsilon \tag{3.17}$$

Let l be the lower bound of the functional V on S_ε. Then, by (3.7), we find

$$l = \inf_{S_\varepsilon} V \geqslant \inf_{S_\varepsilon} [c\|\eta\|^2_{H_\Omega} - m_1\|\eta\|^3_{H_{2\Omega}}] > \frac{c\varepsilon}{2} > 0, \quad \text{if} \quad \varepsilon < \frac{1}{2}\left(\frac{c}{m}\right)^2 \tag{3.18}$$

Further, by analogy with (3.8), we obtain the estimates

$$I_2(\varphi) \leqslant m_5\|\eta\|^2_{H_{2\Omega}}, \qquad I_3(\Phi, \eta) \leqslant m_6\|\eta\|^2_{H_{2\Omega}} \tag{3.19}$$

Now using (3.19), (2.11) and (3.7) we obtain from (3.15) that if $\|\eta_0\|^2_{H_\Omega} < \delta$, then

$$V(t) < V_1(0) < m_7\delta + 2m_1\delta^{3/2} = \delta_1 \tag{3.20}$$

From this (3.4) follows. Indeed, if we assume that a point appears at a certain time t_0 on the sphere S_ε, then

$$V(t_0) > \inf_{S_\varepsilon} V = l > 0$$

Which contradicts (3.20) if $\delta_1 < l$.

4. We shall now consider the case when the second variation is not positive definite. In particular, let the condition

$$\delta^2 J \leqslant -m\|\eta\|^2_{H_{2\Omega}} \qquad (\eta \in E, \ m > 0) \tag{4.1}$$

be fulfilled on a certain space $E \subset H_{2\Omega}$.

(If E coincides with $H_{2\Omega}$, then $\delta^2 J$ has a strong maximum.) Then the stationary solution W is unstable in the Liapunov sense in the space H_Ω.

For proof, we consider the functional and its derivative with respect to t

$$A(t) = \int_\Omega \eta\eta_t \, dx\, dy, \qquad A'(t) = \int_\Omega \eta_t^2 \, dx\, dy + \int_\Omega \eta\eta_{tt} \, dx\, dy \tag{4.2}$$

We shall show that a small number ε can be found such that

$$A'(t) \geqslant m_1 \|\eta\|_{H_\Omega^2} > 0 \qquad (m_1 > 0,\ \eta \in E) \quad \text{if} \quad \|\eta\|_{H_\Omega} < \varepsilon \tag{4.3}$$

Assuming the solution to be in the form

$$w = W + \eta, \qquad F = \Phi + \varphi + \psi \tag{4.4}$$

we derive from non-stationary equations of the problem, the equations of perturbed motion

$$\eta_{tt} + \varepsilon^2\Delta^2\eta - [\eta, \Phi] - [\eta, \varphi + \psi] - [W, \varphi + \psi] = 0 \tag{4.5}$$

$$\Delta^2\varphi + [W, \eta] = 0, \qquad \Delta^2\psi + {}^1/_2\, [\eta, \eta] = 0 \tag{4.6}$$

Here the notation $[a, b] = a_{xx}b_{yy} + a_{yy}b_{xx} - 2a_{xy}b_{xy}$ is used. Using (4.5) and (4.6), we find

$$\int_\Omega \eta\eta_{tt} \, dx\, dy = -2\delta^2 J + R_*(\eta)$$
$$R_*(\eta) = -3\int_\Omega \Delta\varphi\Delta\psi \, dx\, dy - 2\int_\Omega (\Delta\psi)^2 \, dx\, dy \tag{4.7}$$

Using (2.7), (2.9) and (3.10) together with estimates (2.50) from [6], we obtain from (4.7)

$$\|R_*(\eta)\| \leqslant m_2 \|\eta\|_{H2\Omega}^3 \tag{4.8}$$

Now the estimate (4.3) follows from (4.2) and (4.8), provided $\|\eta\|_{H_\Omega} < \varepsilon$ and ε is sufficiently small. Further, let us consider a set of such η for which

$$A_0 = \int_\Omega \eta_0\eta_{0t} \, dx\, dy \geqslant 0, \qquad \eta_0 = \eta|_{t=0} \tag{4.9}$$

By (4.3), for such η $A(t) > 0$. Then the instability of W can be shown using the Liapunov theorem on instability (see, e.q. [9]). Here, the functional A plays the role of the Liapunov function.

5. We shall consider a number of particular cases.

(*a*) In the class of axially-symmetric solutions, the equilibrium of a symmetrically loaded plate under a variety of boundary conditions [10 and 11] is unique and therefore stable. Indeed, from the uniqueness of the solution it follows that a strong minimum is reached on it.

(*b*) It was shown in [13 and 14] that the second variation $\delta^2 J(\eta) \leqslant 0$ (for some η) for symmetrical equilibrium of a circular plate uniformly compressed around the edge by sufficiently large loads, as well as for a rigidly clamped circular plate under uniform pressure. Section 4 shows in what sense these solutions are unstable.

(*c*) The axially symmetric equilibrium of a symmetrically loaded plate rigidly fixed around the edge is a membrane equilibrium [2] and therefore stable (with respect to nonsymmetrical disturbances). In connection with this, it is interesting to note that the solution in this case is not unique [12] and that the absolute minimum of the energy is reached in the state of nonsymmetrical equilibrium.

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